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Scaling functions and scaling exponents in turbulence

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We extend the recent work of Sirovich, Smith, and Yakhot (unpublished) and obtain for structure functions of arbitrary order an expression that is uniformly valid for the dissipation as well as the inertial range of scales. We compare the expression with experimental data obtained in a moderate-Reynolds-number turbulent boundary layer and find good agreement. This enables a more definitive determination of the scaling exponents and intermittency corrections than has been possible in the past. The results are substantiated by several consistency checks.

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It is well known that Kolmogorov's hypotheses of 1941 and 1962 [1,2] lead to the following form for longitudinal structure functions of velocity fluctuations in high-Reynolds-number turbulent flows:

$$S_n(r)=\langle[\Delta u_r]^n\rangle \propto r^{\zeta_n}\ , \quad \eta \ll r \ll L\ . \tag{1}$$

Here, $\Delta u_r=u(x+r)-u(x)$ is the velocity increment over a distance r in the direction of the velocity component u, η is the Kolmogorov length scale representing dissipative scales of turbulence, and L is the integral length scale representing the largest correlated motion. The 1941 prediction that $\zeta_n=n/3$ was modified in 1962, according to Obukhov [3], to reflect corrections arising from the intermittent nature of the energy dissipation rate. Much experimental effort has been expended on the determination of the scaling exponents ζ_n. (For a seminal paper related to this effort, see Ref. [4]; for the most recent summary of all previous work, see Ref. [5].) Nearly all experiments to date have concluded that $\zeta_n<n/3$ (for $n>3$). However, a precise determination of the scaling exponents has been hampered by several operational problems such as the inadequacy of Taylor's frozen-flow hypothesis, the uncertain convergence of high-order structure functions, and the *a priori* decision about the precise scaling region. The issue is not completely resolved also because of another fundamental difficulty: in general, unless the scaling region is extensive, the lack of knowledge of the scaling function precludes an accurate determination of the scaling exponents. By a scaling function, we mean a function that exhibits not only the asymptotic scaling behavior but also the trend towards such a behavior. Ideally, the scaling function should be derivable from the equations of motion, but this seems to have proved elusive so far. In a recent paper, Sirovich, Smith, and Yakhot [6] used matched asymptotic expansions and analyticity assumptions to derive an explicit expression for the second-order structure function. One of our goals is to use a similar strategy for obtaining an expression for structure functions of arbitrary order. A second goal is to fit this expression to the experimental data on scaling functions and determine high-order scaling exponents with greater accuracy than has been possible so far. Several consistency checks will be made.

As mentioned already, Sirovich, Smith, and Yakhot [6] studied the case of the second-order structure function. On using the well-known relation (exactly valid in the high-Reynolds-number limit for locally homogeneous and isotropic turbulence [7],

$$S_3(r)=-\tfrac{4}{5}\epsilon r+6\nu\frac{dS_2(r)}{dr}\ , \tag{2}$$

where the dissipation rate $\epsilon=15\nu\langle(\partial u/\partial x)^2\rangle$ and ν is the kinematic viscosity coefficient, these authors found that

$$S_2(r)=\frac{\langle(\partial u/\partial x)^2\rangle r^2}{\left[1+\dfrac{2}{(2-\zeta_2)\alpha^2}(r/\eta)^2\right]^{(2-\zeta_2)/2}}\ . \tag{3}$$

For high Reynolds number $1/\alpha^2=S/12\sqrt{60}$, S being the skewness of the velocity derivative, given by

$$S=\frac{\langle(\partial u/\partial x)^3\rangle}{\langle(\partial u/\partial x)^2\rangle^{3/2}}\ . \tag{4}$$

To extend this analysis to the high-order structure function, recall the following results [8].

(a) The nth-order structure function can be easily expanded to order $(n+4)$ as

$$S_n(r)=A_n r^n\left[1-D_n\frac{n}{24}r^2\right]+o(r^{n+4})\ , \tag{5}$$

where

$$A_n=\left\langle\left[\frac{\partial u}{\partial x}\right]^n\right\rangle \tag{6}$$

and

$$D_n=\left\langle\frac{\partial^3 u}{\partial x^3}\left[\frac{\partial u}{\partial x}\right]^{n-1}\right\rangle\Big/A_n\ . \tag{7}$$

(b) The constants D_n in Eq. (7) can be expressed as

$$D_n\approx K_n/\eta^2\ , \tag{8}$$

where K_n is a phenomenological constant (to be determined) that is approximately independent of n. The con-

stants K_n can be expected to be such that $K_{\text{even}} < K_{\text{odd}}$ because, in Eq. (7), more cancellations occur on the average for even n than for odd n. Inserting Eq. (8) into Eq. (5), we have

$$S_n = A_n \eta^n \left[\frac{r}{\eta}\right]^n \left[1 - K_n \frac{n}{24}\left[\frac{r}{\eta}\right]^2\right], \tag{9}$$

valid for small r/η. For large r/η, we have from Eq. (1) the expectation that

$$S_n \propto \left[\frac{r}{\eta}\right]^{\zeta_n}. \tag{10}$$

Matching the two expressions (9) and (10), we arrive at the result that

$$S_n(r) = \frac{A_n \eta^n (r/\eta)^n}{\left[1 + K_n \frac{n}{(n-\zeta_n)12}(r/\eta)^2\right]^{(n-\zeta_n)/2}}. \tag{11}$$

For convenience, we rewrite Eq. (11) as

$$S_n(r) = \frac{A_n \eta^n (r/\eta)^n}{[1 + B_n (r/\eta)^2]^{C_n}}. \tag{12}$$

Equation (12) differs from Eq. (11) in that, in the former, we have not prescribed the precise form for B_n or C_n. However, if the reasoning leading to Eq. (11) were correct, we should find

$$B_n C_n \approx \frac{K_n}{24} n, \tag{13}$$

with $K_{2j} \equiv K_{\text{even}} < K_{2j+1} \equiv K_{\text{odd}}$ for $j \geq 1$.

The physical meaning of the parameters A_n, B_n, and C_n in Eq. (12) is quite straightforward. Equation (5) shows that the A_n are the averaged nth power of the longitudinal velocity derivative. It is therefore a dissipation quantity. C_n is an inertial range quantity, from which the scaling exponents ζ_n can be easily computed as $\zeta_n = n - 2C_n$. $B_n = K_n/C_n$ is related through K_n to the Taylor expansion of the structure function at values of $r \sim \eta$, and contains the information about the exponents ζ_n through C_n. It is therefore a mixture between dissipative and inertial properties. In fact, the transition from dissipative to inertial regimes occurs when $B_n (r/\eta)^2 \gg 1$, i.e., at $r/\eta > 1/B_n^{1/2}$. It will be seen later that $1/B_n^{1/2} \approx 11$.

We now wish to study the goodness of fit of Eq. (11) or Eq. (12) with the experimental data on structure functions in both dissipative and inertial ranges. Let us hasten to note, however, that the function

$$S_3 = \frac{A_3 \eta^3 (r/\eta)^3}{1 + (K_3/8)(r/\eta)^2} \tag{14}$$

[with $A_3 = (K_3/10)(\epsilon/\eta^2)$] cannot rigorously be the third-order structure function. In effect, if it were, we would have S_2 given exactly by Eq. (2) as $S_2 = [8(\eta\epsilon)^{2/3}/15K_3] \ln[1 + (K_3/8)(r/\eta)^2]$, which is inconsistent [9] with the anticipated scaling form as well as with Eq. (3) of Sirovich, Smith, and Yakhot. This naturally invites the following question: why should one be interested in an expression for S_3 which yields an inconsistent expression for S_2? First, the present S_3 is numerically indistinguishable from that obtained from Eqs. (2) and (3). Second, neither the S_n given in Eq. (11) nor the S_2 of Sirovich, Smith, and Yakhot is exact, but should be viewed as good approximations to scaling functions. In contrast to the usual expression $S_n \propto r^{\zeta_n}$, which is only asymptotic and misses all the information about dissipative scales, Eq. (11) contains not only the asymptotic limit, but also the manner in which it is approached. Thus, it provides a much better ansatz for the structure functions. This is the spirit in which Eq. (11) must be viewed.

Turbulent velocity data were acquired in a boundary layer on a smooth flat plate in a wind tunnel. The wind tunnel had a cross-sectional area of 72×56 cm^2, and a test section length of 170 cm from the leading edge of the flat plate. The distance between the wind tunnel upper wall and the top surface of the flat plate (on which the measurements were made) was 39 cm. The boundary layer was tripped by a combination of sandpaper strips and a circular rod, and the Reynolds number based on the boundary layer thickness δ was 32 000. The streamwise pressure gradient was adjusted to be negligible. The boundary layer had the expected log-law region with standard log-law constants. More details on the characterization of the boundary layer can be found in [10]. Velocity measurements were made at a height of 0.2δ which is at the edge of the fully turbulent part of the boundary layer. The boundary layer was 114 mm thick, and the flow velocity in the free stream was 12.2 m/s. Velocity data were acquired with a standard hot wire operating on a constant-temperature anemometer. The data file was long (ten million points covering a real-time duration of about 30 min). It was ascertained [5] that structure functions at least up to the fifteenth order converged. The microscale Reynolds number was estimated in [5] to be about 200. Some additional statistical data are listed in Table I. Even though the flow Reynolds number is not large enough to determine scaling exponents unambiguously from the use of Eq. (1), Zubair [5] has made a careful and detailed study, and obtained best estimates of the scaling functions (with which we will be able to make comparisons). The best spatial and temporal resolution of the data is of the order of three Kolmogorov scales, thus missing a significant part of the dissipation region. In spite of this limitation (which we hope to eliminate in the near future), it is thought that a comparison with the theoretical expressions would be beneficial.

Using structure functions computed from these data, we obtained from least-squares fits the parameters A_n, B_n, and C_n. Note that, in contrast to the usual asymptotic scaling fit r^{ζ_n} where both the lower and upper limits of

TABLE I. Additional parameters characterizing the small-scale structure of the boundary layer.

Quantity	Value
S	0.39
K	5.5
$(\epsilon/\nu)^{1/2}$	206 sec^{-1}

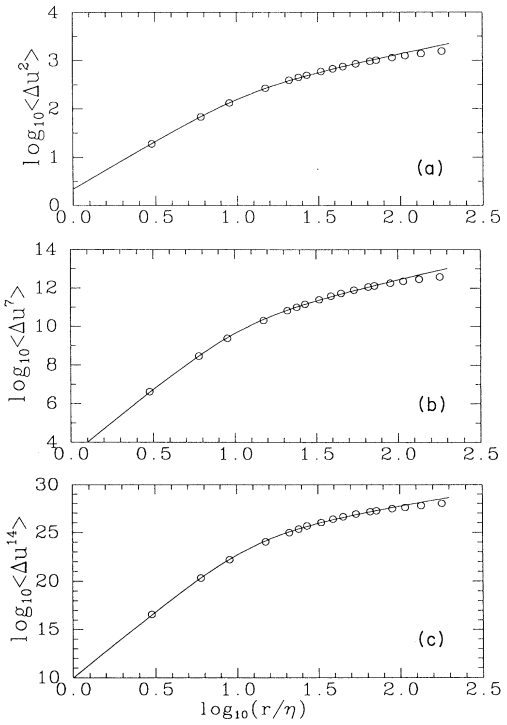

FIG. 1. Second (a), seventh (b), and fourteenth (c) -order structure functions from experimental data. Solid line is the best fit using Eq. (12). In terms of convergence, the uncertainty in the data is comparable to the size of the symbols.

the inertial range had to be assessed, we can fit the data in both dissipative and inertial range—and hence need to be concerned only about the upper limit. Figures 1(a)–1(c) show as examples the second-, seventh-, and fourteenth-order structure functions fitted by Eq. (12). The fits are shown by a solid line. The agreement is good, and comparable for structure functions of all other orders. Table II lists the various constants obtained by such fits for n up to 15. The scaling exponents ζ_n listed in Table II are close to those obtained earlier by Zubair [5]. Figure 2 shows both sets of exponents: those found in this paper (diamonds) and those found in [5] (circles). The error bars are from Zubair [5]. Although the new

TABLE II. The constants A_n, B_n, and C_n, and the scaling exponents ζ_n.

n	A_n [(sec)$^{-n}$]	B_n	C_n	ζ_n
2	22.06×10^{-1}	7.2×10^{-3}	0.65	0.70
3	12.31×10^{-1}	9.8×10^{-3}	1.00	1.00
4	28.28	8.5×10^{-3}	1.40	1.20
5	44.07	8.9×10^{-3}	1.74	1.52
6	73.46×10^{1}	8.3×10^{-3}	2.19	1.62
7	21.10×10^{2}	8.7×10^{-3}	2.52	1.96
8	28.55×10^{3}	7.8×10^{-3}	3.00	2.00
9	13.09×10^{4}	8.5×10^{-3}	3.32	2.36
10	17.39×10^{5}	7.8×10^{-3}	3.82	2.36
11	95.64×10^{5}	8.2×10^{-3}	4.15	2.70
12	12.51×10^{7}	7.5×10^{-3}	4.66	2.68
13	77.04×10^{7}	8.1×10^{-3}	4.96	3.08
14	10.17×10^{9}	7.3×10^{-3}	5.49	3.02
15	66.79×10^{9}	7.8×10^{-3}	5.76	3.48

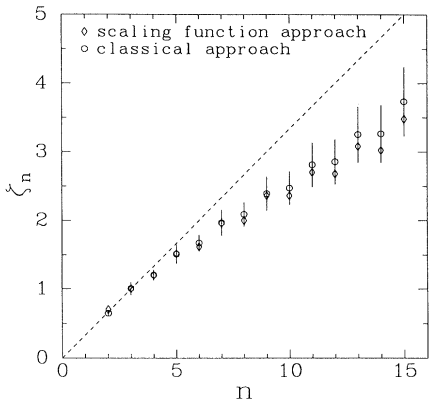

FIG. 2. Plot of the scaling exponents ζ_n versus n computed using the scaling functions (diamonds) and the classical power-law fits (circles) [5]. The dashed line is the $n/3$ prediction of Kolmogorov's 1941 theory.

scaling exponents fall well within the error bars given by [5], it is clear that the circles tend to be above the diamonds for large n. We suspect that this is because the lower bound of the scaling range for high-order structure functions in [5] was underestimated. (The upper bound of the scaling region was the same in both cases.) The present data fully support the conclusions of Zubair [5] that $\zeta_n < n/3$ for $n > 3$, and that the odd exponents are organized on a curve that is distinct and higher than that for even n.

As already remarked, consistency between Eqs. (11) and (12) requires that $B_n C_n = (K_n/24)n$. Figure 3 shows a plot of $24B_nC_n/n$ as a function of n. We can see that, except for $n=2$ and possibly $n=3$, the K_n are constant and independent of n, different for odd and even n: $K_{\text{even}}=0.071$ (dotted line) $< K_{\text{odd}}=0.074$ (dashed line). This difference is small but significant, as explained in the caption to the figure.

At this point several tests of consistency can be made. First, consider the skewness of the velocity derivative. From Eqs. (4) and (5) and Table II, we have

$$S=\frac{A_3}{A_2^{3/2}}=0.38\ , \tag{15}$$

which is in good agreement with the measured value given in Table I. Alternatively, as mentioned earlier, the

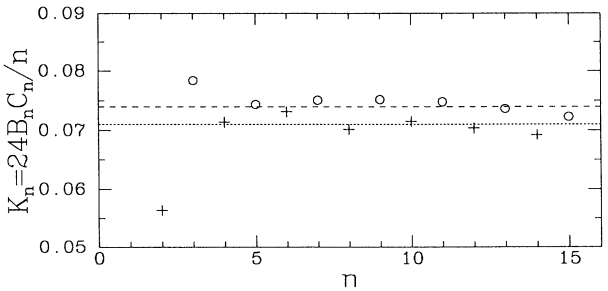

FIG. 3. Plot of $K_n = 24B_nC_n/n$ versus n for even (plus) and odd values of n (circles). For both even and odd values of n separately, the standard deviations σ_{even} and σ_{odd} are each 0.001. The mean and standard deviations for all n taken together are 0.0726 and 0.002, respectively. The statement that $K_{\text{even}} < K_{\text{odd}}$ is statistically significant because $K_{\text{even}} + \sigma_{\text{even}} < K_{\text{odd}} - \sigma_{\text{odd}}$.

skewness can also be computed through

$$S=\sqrt{60}K_2=(\sqrt{60})12B_2C_2=0.43\ . \tag{16}$$

This is again in reasonable agreement with measurements.

Second, consider the kurtosis of the velocity derivative $K=\langle(\partial u/\partial x)^4\rangle/\langle(\partial u/\partial x)^2\rangle^2$. From Eqs. (4), (5), and Table II, we find that $K=A_4/A_2^2=5.8$, in good agreement with direct measurements (see Table I).

Third, from the theory of locally isotropic turbulence [1,11], we know that $\langle(\partial u/\partial x)^2\rangle=\epsilon/15\nu$ and that for $r/\eta\gg1$, $S_3/(r/\eta)=-\frac{4}{5}\epsilon\eta$. It follows that

$$(15A_2)^{1/2}=\left[\frac{\epsilon}{\nu}\right]^{1/2}, \tag{17}$$

$$\left[\frac{5}{4}\frac{A_3}{B_3}\right]^{1/3}=\left[\frac{\epsilon}{\nu}\right]^{1/2}. \tag{18}$$

From Table II we find (in units of $\sec^{-1}$)

$$(15A_2)^{1/2}=213\ , \tag{19}$$

$$\left[\frac{5}{4}\frac{A_3}{B_3}\right]^{1/3}=200\ . \tag{20}$$

There is a relative difference of less than 7% between the two estimates for $(\epsilon/\nu)^{1/2}$, again in excellent agreement with measurements (see Table I).

Finally, it is interesting to note that the asymptotic limit $r/\eta\gg1$ of the second-order structure function

$$S_2=\frac{A_2\eta^2}{\left[\dfrac{S}{(2-\zeta_2)12\sqrt{15}}\right]^{(2-\zeta_2)/2}}(r/\eta)^{\zeta_2} \tag{21}$$

(where we have used $K_2=S/\sqrt{60}$) can be expressed in the more familiar form as

$$S_2=C_2(L\epsilon)^{2/3}(r/L)^{\zeta_2}\ . \tag{22}$$

This allows us to write the skewness S in terms of the constant C_2 and the microscale Reynolds number Re_λ as

$$S=\left[\frac{A^{4/3}}{15}\right]^{(3/2)(\zeta_2-2/3)/(2-\zeta_2)}\times\frac{12\sqrt{15}(2-\zeta_2)}{(15C_2)^{2/(2-\zeta_2)}}\mathrm{Re}_\lambda^{3(\zeta_2-2/3)/(2-\zeta_2)}, \tag{23}$$

where A is a constant of order unity defined by $A=\epsilon L/u^3$. Equation (22) shows that S varies as the $3(\zeta_2-2/3)/2-\zeta_2$ power of Re_λ. Under a completely different approach [12] in the context of generating turbulencelike signals, the same result was found analytically. In Kolmogorov's 1941 theory [1] in which $\zeta_2=\frac{2}{3}$, one finds that $C_2=(16/15S)^{2/3}$. This relation emphasizes the well-known fact [11] that the inertial range constant C_2 (universal in the context of [1]) is related to the nominally dissipative quantity S, which makes the latter quantity special. Using the experimental value for S from Table I, we obtain $C_2=1.96$, which falls within the accepted range [11] of 1.8–2.2. If intermittency is taken into account in the form of $\zeta_2=0.71$[13] and A is taken to be unity [14], one has from Eq. (23) the result that

$$S=0.27\mathrm{Re}_\lambda^{0.1}\ . \tag{24}$$

Equation (24) is in reasonable agreement with experimental data (see, for example, [15]).

In conclusion, this work emphasizes that expressions for structure functions uniformly valid over the dissipative and inertial ranges can be found. They contain only a few parameters with easily identifiable interpretation, and capture a great deal of quantitative information. Finally, it shows that the deviations of the scaling exponents from the classical values $n/3$ are not an artifact of the small size of the scaling region.

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